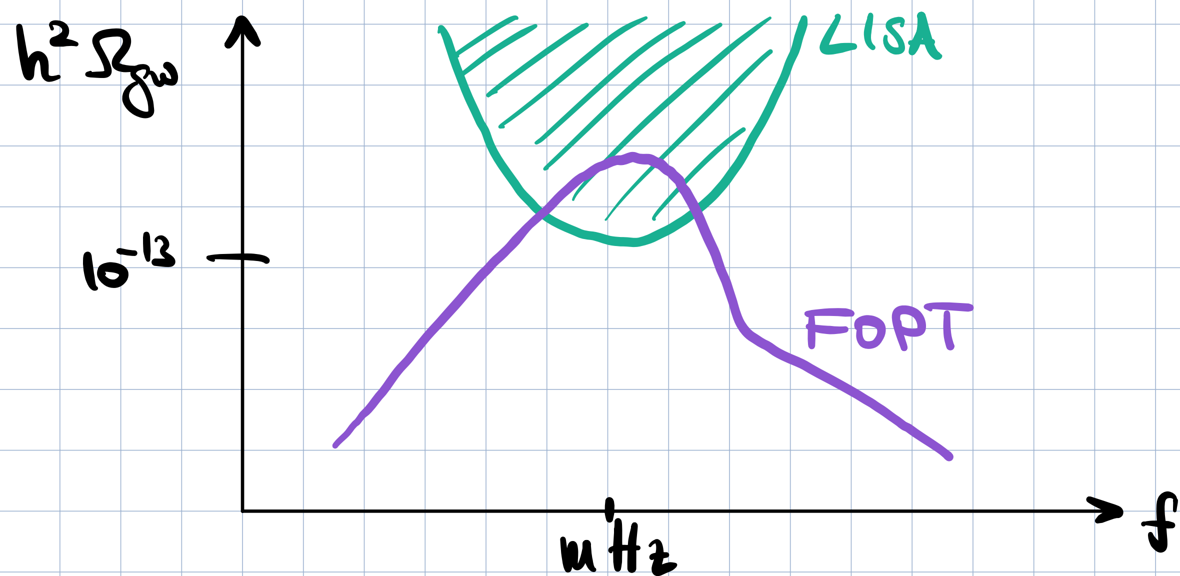
Lecture notes Gravitational wave cosmology

* **The homogeneous universe**
  - FLRW metric
  - cosmic pie chart
  - chronology of the universe
* **The perturbed universe**
  - GWs in flat space & vacuum
  - GWs in an FLRW background
  - stochastic GW backgrounds
* **Phase transitions**
  - First-order phase transitions
  - QFT @ finite temperature
  - Bubble nucleation & percolation
  - GW spectra from FOPTs

28.04.2025

Goal: understand this plot



§1 The homogeneous universe

$$G_{\mu\nu}(x) = \frac{1}{m_{pl}^2} T_{\mu\nu}(x)$$

Einstein tensor, spacetime curvature:
expansion of universe, GWs, ...
function of $g_{\mu\nu}$ & derivatives

metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Planck mass,
$m_{pl} = (8\pi G)^{-1/2}$
$= 2.4 \cdot 10^{18}$ GeV

Energy momentum tensor;
sourced by QFT: energy, masses, motion in plasma, ...

* Friedmann-Lemaître-Robertson-Walker metric

→ Assume isotropy & homogeneity & perfect fluid ("hot plasma")

$$\underset{\text{„Hubble rate"}}{H} \equiv \frac{\dot{a}}{\underset{\text{"scale factor"}}{a}} = \sqrt{\frac{\rho}{3m_{PL}^2}}$$

energy density of plasma

and

pressure of plasma

$$\dot{\rho} = -3H(\rho + P)$$

"Matter tells space how to curve and space tells matter how to move"

Here: Plasma energy density ρ tells universe how fast to expand (H) and the expansion of space tells matter how to dilute.

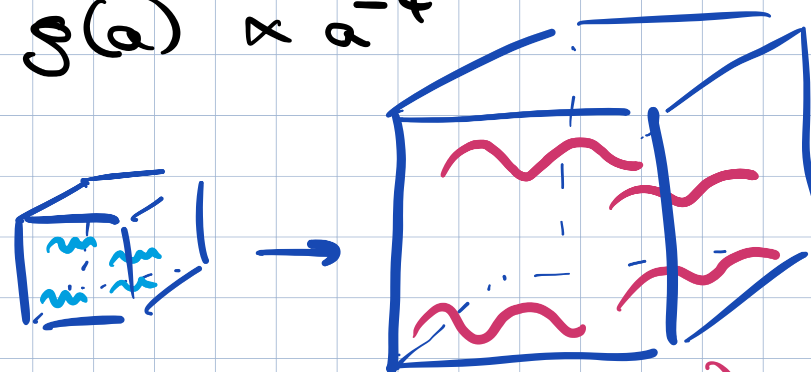
→ for dust : $P = 0$

$$\rightsquigarrow \rho(a) \propto a^{-3}$$

$\rightsquigarrow \quad \rho(t) \propto t^{-2/3}$

→ for radiation (= hot plasma): $P = \frac{\rho}{3}$

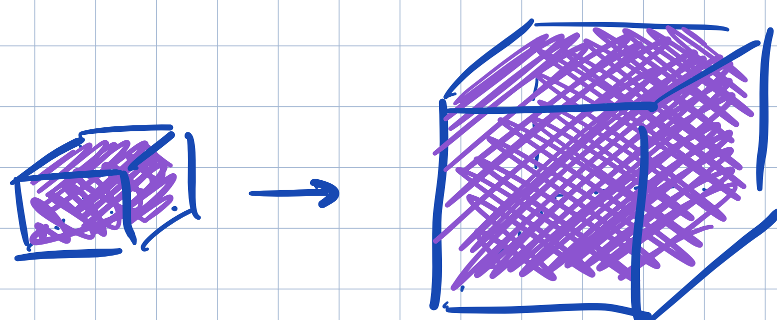
$\rightsquigarrow \quad \rho(a) \propto a^{-4}$



radiation (e.g. photons) get diluted & redshifted

$\rightsquigarrow \quad \rho(t) \propto t^{-1/2}$

→ vacuum energy $\quad P = -\rho$

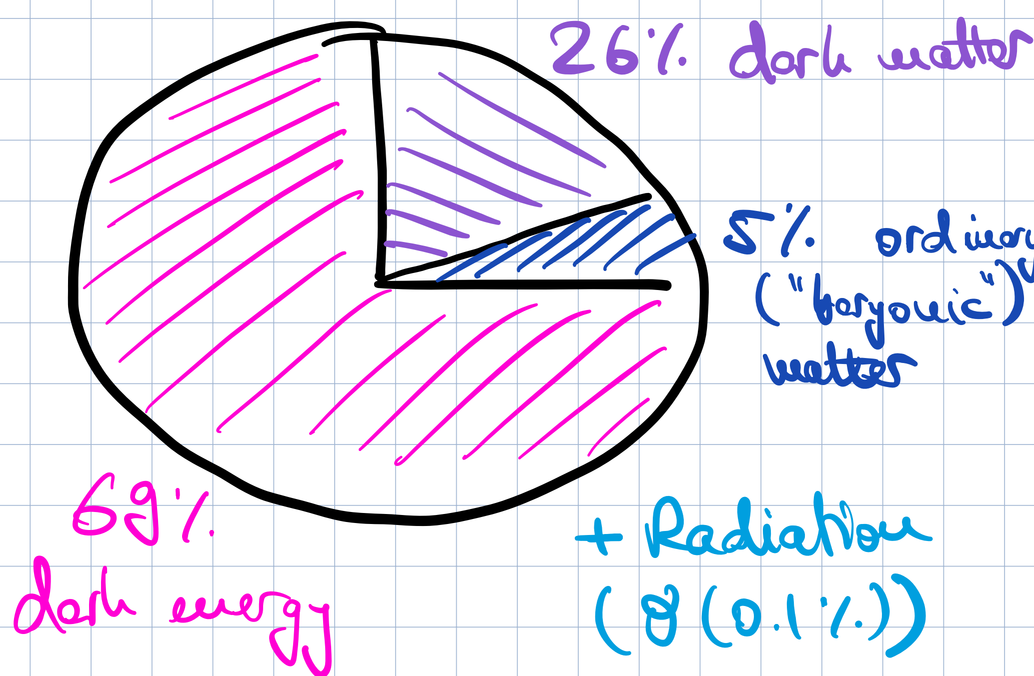


$\rightsquigarrow$ ρ independent of a & t

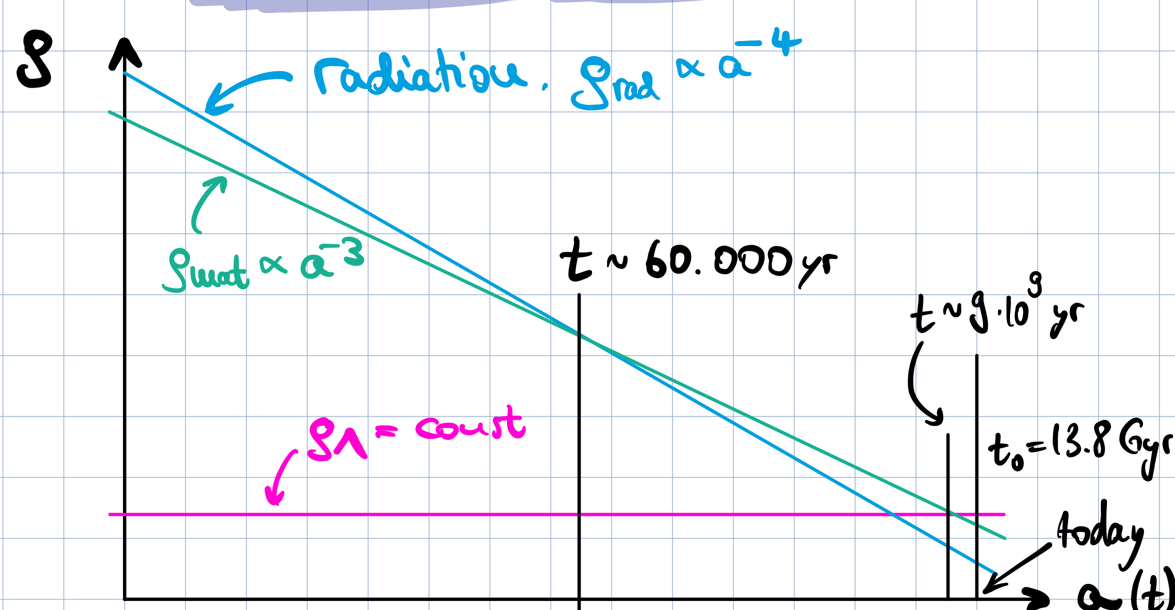
$\rightsquigarrow$ makes space expand like
$a(t) \propto \exp(Ht)$

with $H = \text{const.}$

* Universe today



* Can solve Friedmann eq. "backwards in time"



$$\underbrace{\qquad\qquad}_{\text{Radiation domination}}\underbrace{\qquad\qquad}_{\text{Matter domination}}\sim$$

Radiation domination — Matter domination — dark energy domination

§2 The perturbed universe

$$G_{\mu\nu} = \frac{1}{m_{Pl}^2} T_{\mu\nu}$$

what if $\rho(\vec{x}) \neq$ const?

* GWs in flat space:

Linearize Einstein's equation and find that

→ small metric perturbations $h_{\mu\nu}(\vec{x},t)$ behave like a wave (i.e, they propagate) in vacuum:

$$\Box\, h_{\mu\nu}^{TT} = \left(\partial_t^2 - \frac{1}{c^2}\Delta\right) h_{\mu\nu}^{TT} = 0$$

(TT: "transverse-traceless" gauge, a choice of the coordinate system

* The effect of GWs on test masses

$$\Box h^{TT}_{\mu\nu} = 0$$

- Assume plane wave moving in $\vec{k}$ - direction:

$$h^{TT}_{ij}(x) = \mathrm{Re}\left[e_{ij}(\vec{k})\, e^{ikx} \right]$$

- GW moves with $c=1$ ~ dispersion relation $\omega \equiv k_0 = |\vec{k}| \Rightarrow k_\mu = (\omega, \vec{k})$
- Assume $\vec{k} \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ without loss of gen.
  $\Rightarrow \vec{k} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$.

- We work in a frame of reference in which the TT gauge holds:

$$h^{0\mu} = h^i{}_i = \partial^i h_{ij} = 0 .$$

$$\hookrightarrow k^i h_{ij} = 0$$

$$\Rightarrow 0 + 0 + \omega h_{3j} = 0$$

$$\Rightarrow h_{3j} = h_{j3} = 0$$

$$h^{TT}_{\mu\nu} = \begin{pmatrix} \cancel{h_{00}} & \cancel{h_{10}} & \cancel{h_{20}} & \cancel{h_{30}} \\ \cancel{h_{01}} & h_{11} & h_{21} & \cancel{h_{31}} \\ \cancel{h_{02}} & h_{12} & h_{22} & \cancel{h_{32}} \\ \cancel{h_{03}} & \cancel{h_{13}} & \cancel{h_{23}} & \cancel{h_{33}} \end{pmatrix}$$

$$h_{11} + h_{22} = 0$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-z)]$$

$\rightsquigarrow$ there are two propagating degrees of freedom,

$+$ & $\times$ polarization.

- The corresponding infinitesimal line element reads

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu$$

$$= dt^2 - dx^2 \{1 + h_+ \cos(\omega(t-z))\}$$
$$- dy^2 \{1 - h_+ \cos(\omega(t-z))\}$$
$$- 2\,dx\,dy\, h_\times \cos(\omega(t-z))$$
$$- dz^2 .$$

- Consider two test masses @ $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$

$\Rightarrow$ Coordinate distance in x-direction $L_x = x_1 - x_2 = \text{const}.$

but proper distance:

$$s = L_x \sqrt{1 + h_+ \cos(\omega t)}$$
$$\approx L_x \tfrac{1}{2} h_+ \cos(\omega t).$$

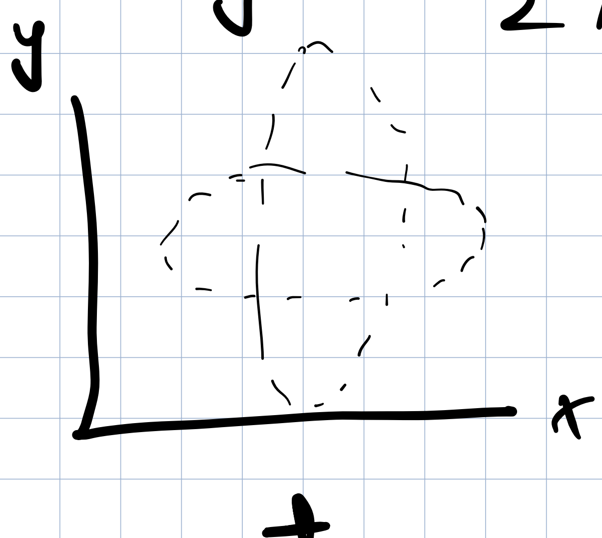
- If two test masses were mirrors reflecting a light beam hence and forth, the proper distance would correspond to the runtime of light, which would oscillate.

$\Rightarrow$ Working principle of interferometers & PTAs.

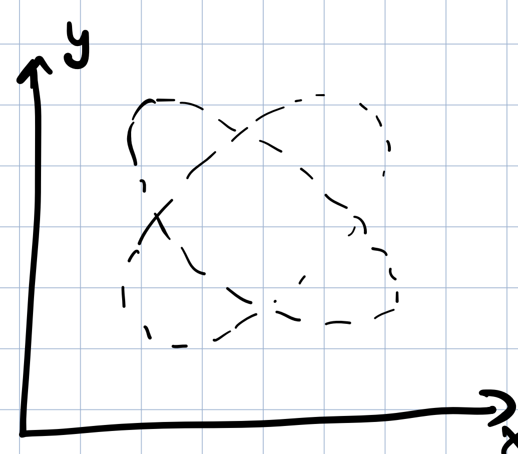
– Now consider two test masses at $(t,0,0,0)$ and $(t, x_i, y_i, 0)$. A $\overset{\times}{+}$ - polarized wave causes shift

$$\delta s_x \simeq \frac{h_+^{\times}}{2}\, \overset{y_i}{x_i} \cos(\omega t),$$

$$\delta s_y \simeq \overset{+}{-} \frac{h_+^{\times}}{2}\, \overset{\times}{y_i} \cos(\omega t)$$



$+$



$\times$

→ They are sourced by anisotropic stress:

far away from source, distance r

$$h_{ij}^{TT}(t,\vec{x}) \sim \frac{1}{r} \frac{1}{M_{Pl}^2} \ddot{Q}_{ij}^{TT}$$

with quadrupole moment

$$Q_{ij} = \int d^3x \, \rho(t,\vec{x}) \left[x_i x_j - \frac{r^2}{3} \delta_{ij} \right]$$

("no scalar or dipole source of GWs")

→ direct consequence: A single bubble cannot emit gravitational waves due to spherical symmetry. Need to break isotropy: at least two colliding bubbles.

* GWs in FLRW background

- Solution of Einstein equation becomes more complicated...

- End up with result that there are still metric perturbations, that propagate. Now, they however also get redshifted (like light) to larger wavelengths & smaller amplitudes.

* Stochastic gravitational wave backgrounds:

- At a given time $R_H = \frac{1}{H(t)}$ tells the distance up to which two events can have possibly been in causal contact.

  $\rightsquigarrow$ Correlation lengths between GW sources (e.g. bubbles) need to be "sub-Hubble"

  $$\ell < 1/H(t_*)$$

  at the time of the GW emission t_*.

  $\rightsquigarrow$ Redshifting to today yields

  $$l_0 / R_0 < 10^{-11} \left(\frac{\text{GeV}}{T_*} \right)$$

correlation length, redshifted to today

size of observable universe today

temperature of the universe at the time of the production of the GWs

$\Rightarrow$ The correlation lengths are tiny! A GW produced in the early universe is sourced by incredibly many independent "Hubble patches". E.g. for $T_* \sim 100$ GeV (EWPT), a GW signal comes from at least 10^{24} uncorrelated regions of the celestial sphere

$\Rightarrow$ A cosmological GW background will only be observable as "background noise", described by a statistical distribution of tensor fluctuations $h_{\mu\nu}^{TT}(\vec{x}, t)$.

- Assuming stationarity, gaussianity, isotropy & no polarization a cosmological background is fully described by

$$\Omega_{gw} = \int_0^\infty \frac{df}{f} \underbrace{\frac{1}{\rho_c^0} \frac{d\rho_{gw}^0}{d\log f}}_{\Omega_{gw}(f)}$$

with $\rho_c^0 \equiv 3 m_{Pl}^2 H_0^2$

and $\frac{d\rho_{gw}^0}{d\log f}$: Energy density in GWs today, in a logarithmic frequency bin.

- This definition is analogous to

$$\Omega_\Lambda = \frac{\rho_\Lambda^0}{\rho_c^0} = 0.69$$

$$\Omega_{DM} = \frac{\rho_{DM}^0}{\rho_c^0} = 0.26$$

$$\Omega_b = \frac{\rho_{bar}^0}{\rho_c^0} = 0.05$$

with $\Omega_{tot} = \sum_i \Omega_i = 1$.

$$\Omega_{rad} = \Omega_\gamma + \Omega_\nu + \Omega_{gw} \simeq \mathcal{O}(10^{-5})$$

$\rightsquigarrow$ Gravitational waves contribute only marginally to the energy of a given volume in space.

- There's the so-called "Hubble tension": $H_0 = 73 \frac{km}{s\,Mpc} - 68 \frac{km}{s\,Mpc}$

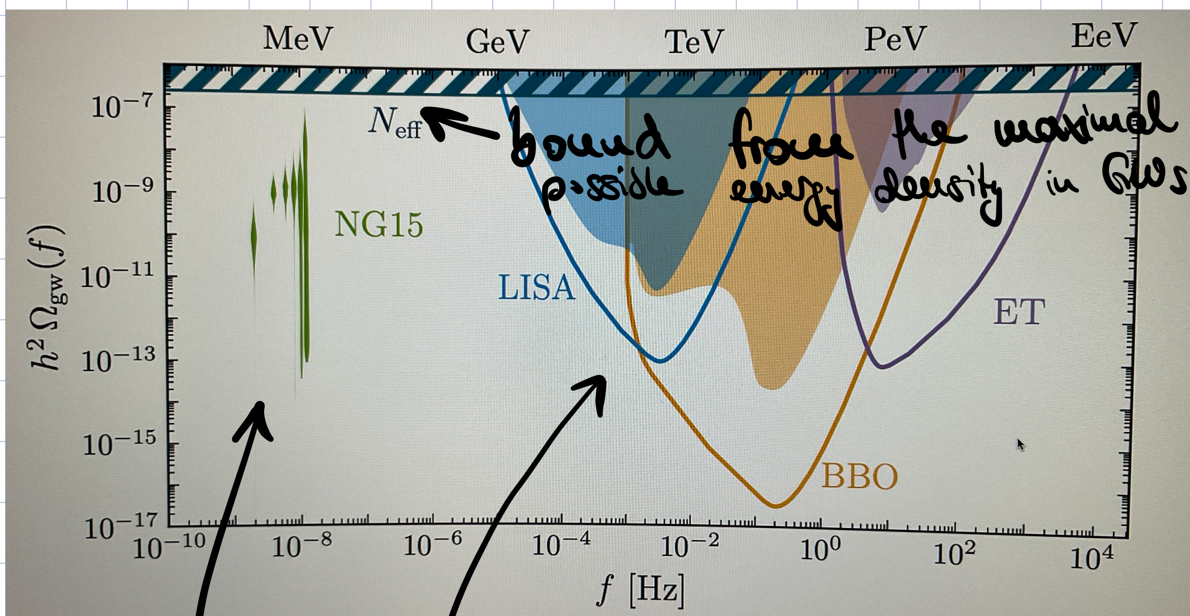
(> 5σ difference between different measurements)

$\rightsquigarrow$ this can be factored out by only computing

$$h^2 \Omega_{gw}(f) \quad \text{with} \quad h = \frac{H_0}{100\,km/s\,Mpc} \simeq 0.7$$

instead of $\Omega_{gw}(f)$. This makes our theory predictions independent of the precise value of the current expansion velocity.

Minimum temperature of the universe at the emission of given GW frequency



bound from the maximal possible energy density in GWs

first GW background observed

frequency of GW today.

LISA, ET & BBO are future GW observatories

§3 Phase transitions

* First-order phase transitions

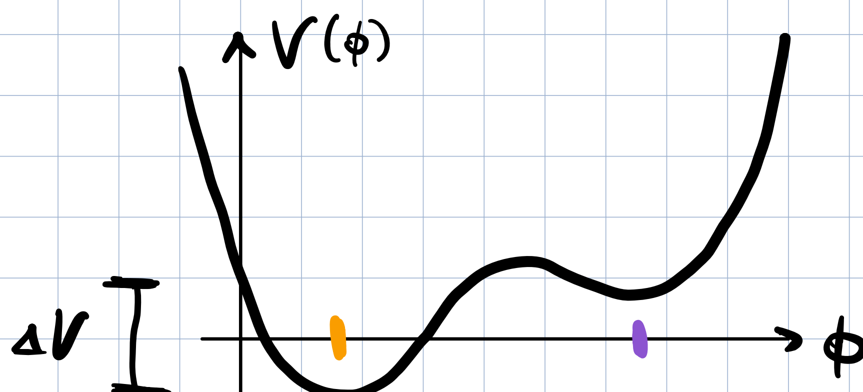
- Can split scalar fields into homogeneous background part & dynamical "quantum fluctuation" part.

- To obtain equations of motion for homogeneous part ϕ, extremize action of "static" scalar field

$$S = \int d^4x \left[\cancelto{0}{\tfrac{1}{2}(\partial_\mu \phi)^2} - V(\phi) \right]$$

$$\rightsquigarrow \quad \delta S = 0 \Leftrightarrow \frac{\partial V}{\partial \phi} = 0 .$$

$\rightsquigarrow$ minima of $V(\phi)$ are the "vacuum expectation value".



true minimum false minimum

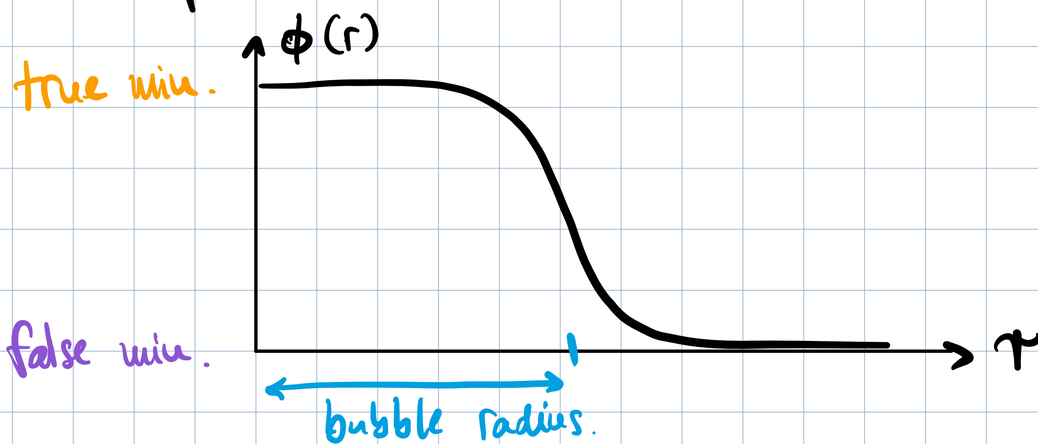
- It can happen that the background field is in the false minimum even though the true minimum would be energetically favored ($\Delta V > 0$).
- Then, the field won't be homogeneous, but will rather follow the Klein-Gordon eq.

$$\Delta \phi = \frac{dV}{d\phi} .$$

- One solution of this is the spherically symmetric bounce

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{3}{r} \frac{\partial \phi}{\partial r} = \frac{dV}{d\phi}$$

which is solved by a field profile in form of a bubble



- Since energy gets liberated ($\Delta V > 0$), the bubbles expand if the pressure of the plasma on the bubble walls is small enough.

$\Rightarrow$ bubbles expand, collide & fill the universe with a new phase.

- If the potential has no barrier, the field simply "rolls down" to its energetically favored vev everywhere in the Universe @ the same time $\rightarrow$ no bubbles, no GWs.

* QFT at finite temperature

- At $T=0$, a given tree-level potential, e.g. $V_{tree} = \mu^2\phi^2 + \lambda\phi^4$, receives quantum corrections. Their effect can be included by promoting V_{tree} to V_{eff}. At 1-loop:

$$V_{eff}(\phi, T=0) = V_{tree} + V_{CW}.$$

with $V_{CW} \sim \int \frac{d^4 k_E}{(2\pi)^4} \ln\left(k_E^2 + m_\phi^2\right)$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{E_\phi}{2} \quad \text{with } E_\phi^2 = k^2 + m_\phi^2$$

- The effect of a plasma interacting with the background field can be described by "rolling up" the imaginary time direction on a cylinder with radius $\frac{1}{T}$:

$$V_{eff}(\phi,T) = V_{tree} + \underbrace{\sum\!\!\!\!\!\!\int \ln\left(k^2+m^2\right)}$$

$$= \frac{T}{2}\sum_n \int \frac{d^3k}{(2\pi)^3} \left(k^2+\omega_n^2+m^2\right)$$

$$= \int \frac{d^3k}{(2\pi)^3}\left[\frac{E_\phi}{2} + T\ln\left(1-e^{-E_\phi/T}\right)\right]$$

$$= V_{CW} + V_T$$

- Effectively, the potential obtains an extra contribution V_T.

Roughly (at high $T \gg m_\phi$):

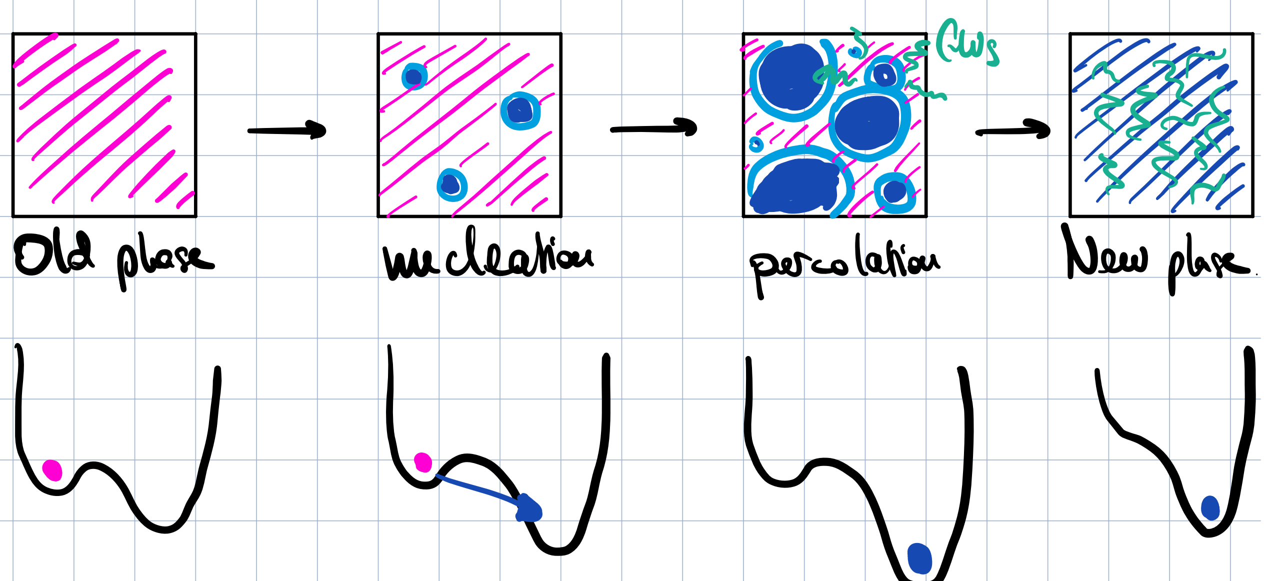
$$V_T = -\frac{\pi^4}{g_0} T^4 + \frac{m_\phi^2 T^2}{24} - \frac{m_\phi^3 T}{12\pi}$$
$$- \# \, m_\phi^4 \log\left(\frac{m_\phi^2}{T^2}\right) + \mathcal{O}(m_\phi^6/T^2)$$

- Origin of the Stefan–Boltzmann law for black-body radiation
  constant offset → no influence on transition
- Restores symmetry @ high temperature: $\propto \phi^2 T^2$
- Generates a ϕ^3 term between the ϕ^2 and ϕ^4 contributions ⇝ relevant for generation of a barrier

between phases.

- If there is a cancellation of the first few terms, this term can also generate barriers which only evolve slowly with temperature.

* Bubble nucleation & percolation



- ≈ 1 bubble per Hubble patch
- ≈ 70% are filled with new phase

- The bubble nucleation rate in a thermal plasma is roughly given by

$$\Gamma \sim T^4 \exp[-S_3/T]$$

where S_3 is the bounce action, i.e. the action of the 3D field theory (the temperature / time direction is "rolled up")

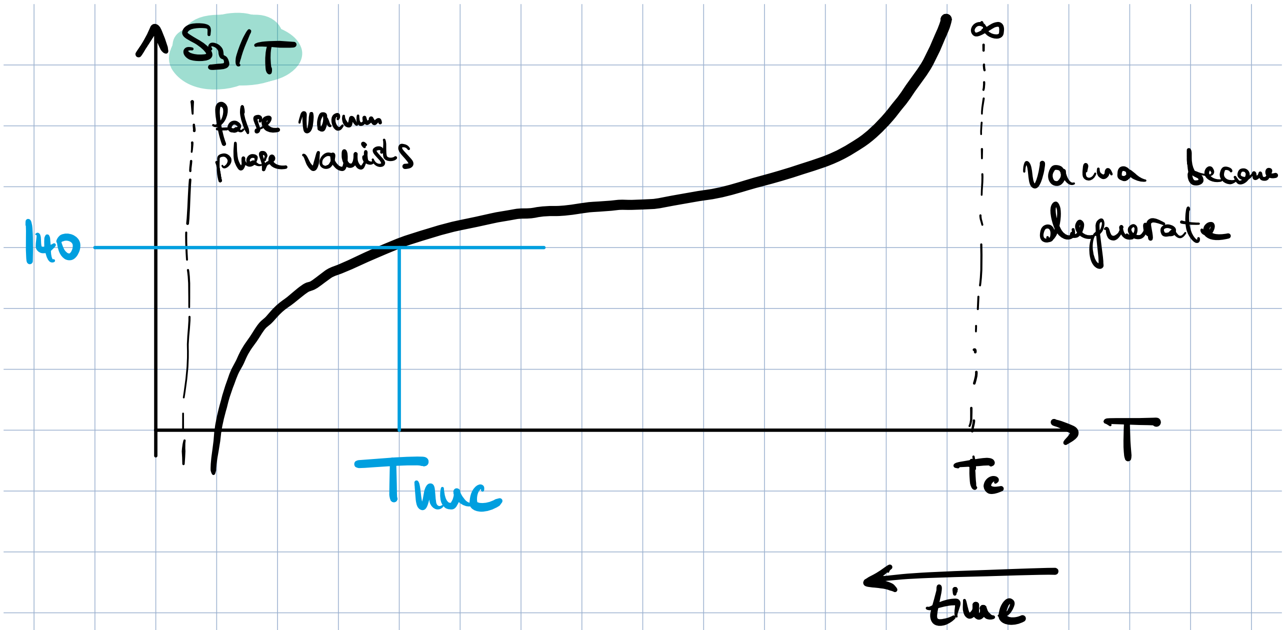
$$S_3 = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$$

+ due to Wick rotation

with ϕ being the bounce solution field profile.

- Nucleation happens when

$$\Gamma \sim H^4 \Rightarrow \frac{S_3}{T} \sim 140$$



- Percolation happens when $I(T_p) = 0.34$ with

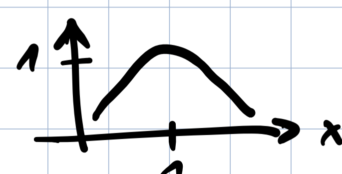
$$I(T) = \frac{4\pi}{3} v_w^3 \int_T^{T_c} dT' \frac{\Gamma(T')}{T'^4 H(T')} \times \left[\int_T^{T''} \frac{dT''}{H(T'')} \right]^3$$

which describes the volume fraction which has already transitioned to the new phase. This equation can only be solved numerically for $T_p \lesssim T_{nuc}$.

* GWs from FOPTs

- There are mainly three parameters influencing the GW signal emitted by a FOPT:

$$h^2 \Omega_{gw}(f) \simeq 10^{-6} \left(\frac{\alpha}{1+\alpha}\right)^2 \frac{1}{\beta/H} \; s(f/f_{p,0})$$

with $s(x) \simeq$ 

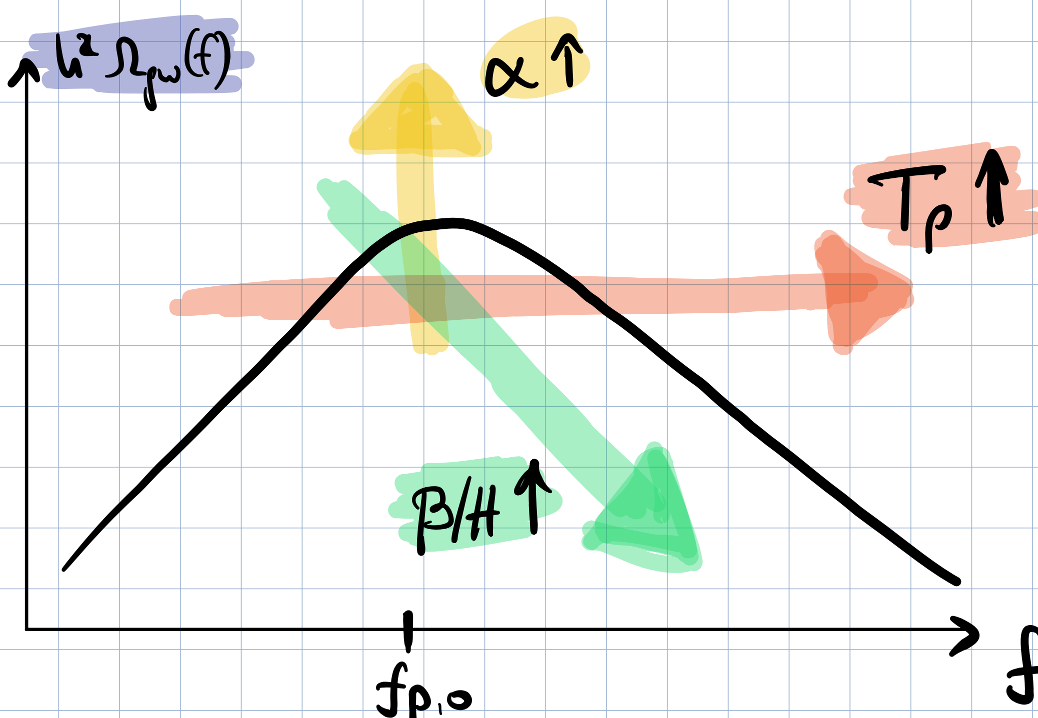
$$f_{p,0} \simeq 10 \text{ mHz} \left(\frac{\beta/H}{100}\right)\left(\frac{T_p}{1 \text{ TeV}}\right)$$

T_p : main influence is the amount of frequency redshift

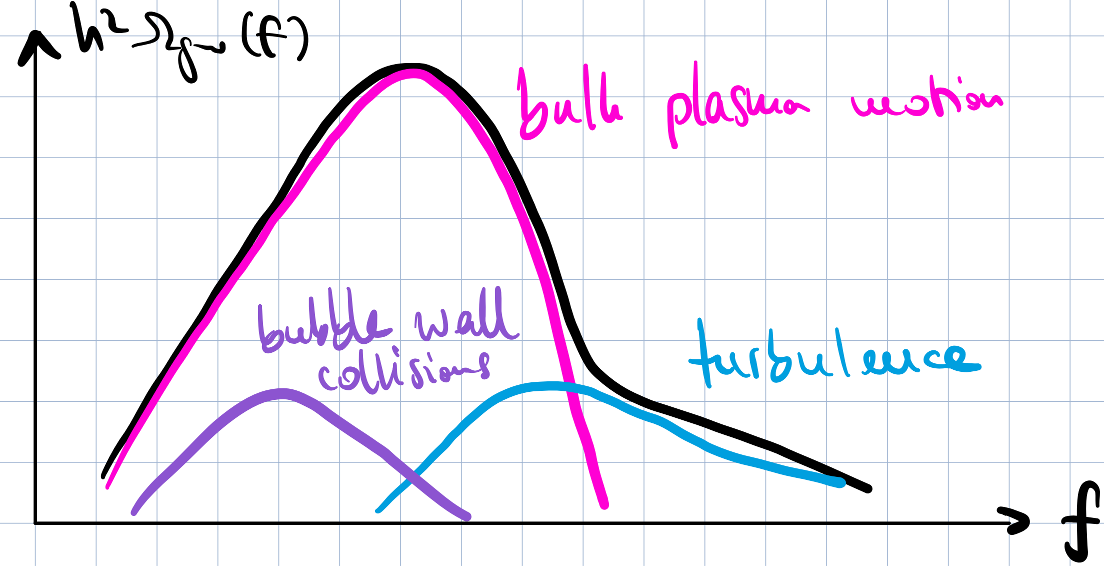
α : roughly $\frac{\Delta V}{\rho_{plasma}}$. The bigger, the stronger the transition, the stronger the GWs

β/H : Speed of the transition (in units of the age^{-1} of the universe at the time of T_p).

The slower the transition, the less bubbles, the bigger the bubbles at collision, the stronger the GW signal and the smaller the GW frequency.



- There are several, independent processes (bubble collisions, bulk plasma motion, turbulence in the plasma) each emitting GWs. The details depend on the bubble wall velocity, which is hard to calculate. Each process has a characteristic spectral shape $s(f/f_{p,0})$ with a distinct peak frequency $f_{p,0}$. The total signal therefore often looks like this



- For FOPTs in which gauge bosons obtain a mass, and if $\alpha \sim \mathcal{O}(1)$, $v_w \to 1$ and bulk plasma motion ("sound waves") dominate the GW signal.